Journal of Statistical Physics, Vol. 69, Nos. 1/2, 1992

A Vortex Model with Superfluid and Turbulent Percolation

Alexandre Joel Chorin[1,2]

Received December 16, 1992

Simple lattice vortex models are presented that exhibit vortex percolation along lines in a temperature/chemical potential plane. Parts of these lines can be identified with the percolation threshold at the inertial range of turbulence, and other parts are analogous to the transition in a three-dimensional XY model that may model the λ transition in superfluidity. Flory exponents at percolation are calculated; for nonnegative temperatures, their values approximate the standard Flory value, and are approximately constant along the transition lines, in agreement with recent conjectures. Conclusions regarding coherent structures in turbulence are also reached.

KEY WORDS: Vortex models; XY model; turbulence; superfluids; Flory exponent.

1. INTRODUCTION

Percolation phenomena have recently been identified in models of turbulence in classical fluids,[1] as well as in models of the critical transition in the three-dimensional XY model,[2,3] which may be a reasonable description of the λ transition in superfluidity. These percolation phenomena have some elements in common as well as some differences. The understanding of their mutual relation can contribute to the understanding of turbulence in both classical and quantum fluids. In the present paper we wish to contribute to that understanding by analyzing a simple vortex lattice model that exhibits a continuum of percolation states, some of which can be identified with turbulent percolation and some with superfluid percolation.

[1] Institute for Advanced Study, Princeton, New Jersey 08540.
[2] Permanent address: Department of Mathematics, University of California, Berkeley, California 94720.

0022-4715/92/1000-0067$06.50/0 © 1992 Plenum Publishing Corporation

In the classical case, one considers a dilute "suspension" of vortex filaments, in thermal equilibrium, with the probability of a configuration being proportional to $\exp(-E/T)$, where E is the energy (the exact expression is given below) and T is a temperature. T can be positive or negative, as in two-dimensional vortex dynamics.[4] For T negative, the vortex filaments are smooth and the energy spectrum $E(k)$ decays rapidly as the wave number k increases. For $T = \pm\infty$ (the sign is immaterial) the energy spectrum has the Kolmogorov form $E(k) \sim k^{-\alpha}$, $\alpha \sim 5/3$, and the vortex filaments have the structure of equal-probability self-avoiding random walks ("polymers"), characterized by the Flory exponent θ,

$$\langle r_L \rangle \sim L^{\theta}, \qquad \text{large } L$$

where L is the length of a vortex strand measured in some appropriate way along its spine, r_L is the end-to-end length of the strand measured by a straight ruler, and $\langle \cdots \rangle$ denotes the appropriate average[5]; $\theta \cong 0.59$; $D = 1/\theta$ is the fractal dimension of the filament. The Kolmogorov exponent α is related to θ and to another exponent at $T = \infty$. For $T > 0$, the vortex filaments are short, and the energy spectrum has the "equipartition" form $E(k) \sim k^2$ (for a definition, see, e.g., ref. 6).

In the evolution of a flow, vortex filaments start out by being smooth, with $T < 0$. Vortex stretching lowers their temperature until $T = \infty$ is reached. This is the maximum entropy state. For Euler flow, the "polymeric" $T = \infty$ state is an uncrossable barrier. In the presence of viscosity, or in underresolved or truncated numerical approximations, the barrier can be crossed; the $T > 0$ equilibria correspond to flows unconstrained by conservation of circulation and connectivity; they can exist physically in scales larger than the scale of the vortical structures in the flow. The polymeric case is a percolation threshold; on one side one has small vortex loops, on the other one has smooth vortex loops; at $T = \infty$ one can have long, fractal vortex filaments. For more detail, see refs. 1, 7, and 8.

A different analysis of the relation between θ, α, and percolation has been suggested by Bershadski,[9] with very similar quantitative conclusions.

In the XY model, a related argument has been made.[2,3] In analogy with the two-dimensional Kosterlitz–Thouless transition mechanism,[10] one assumes that in the superfluid phase, vortex filaments are small; as the transition to a classical fluid state is approached, they can become larger, and at the critical temperature they can become infinitely long, i.e., vortex percolation (defined below) occurs. The temperature T_c of this λ transition is related to the energy of elementary vortex loops, and the assumption that, at the transition, vortex filaments have a polymeric structure yields critical exponents consistent with experiment. The expression for the energy is the same as in the turbulent case. However, the temperature of the λ transition is small and positive, and thus far from infinite. One thus seems to have two different critical points, one at a low and one at a high temperature, with similar geometric properties.

It is important to understand the relation between these percolation thresholds. Such understanding would greatly add to the understanding of superfluid turbulence, and would have a strong bearing on the applicability of real-space renormalization group techniques, such as the Kosterlitz–Thouless analysis, to vortices in classical turbulence. In the present paper, we examine this relation in a simplified lattice vortex model that exhibits both types of percolation. The model also leads to interesting conclusions regarding screening in classical turbulence and the role of coherent structures. It suggests ways of analyzing dense suspensions of vortices and of polymers.

For the sake of clarity, we begin with a very simple model of turbulent percolation that has some interesting features and serves as an introduction to the main model.

2. AN INDEPENDENT-LOOP MODEL

Consider a two-dimensional square lattice, of bond length 1. A lattice site has coordinates (i, j), $i, j \in Z$; a typical square $A_{i,j}$ has sides that connect (i, j), $(i+1, j)$, $(i+1, j+1)$, $(i, j+1)$, (i, j). An elementary "microscopic" vortex loop coincides with the sides of a typical square and is oriented clockwise (Fig. 1). If two adjoining squares contain vortex loops, the vortex lines on their common edge cancel and one obtains a longer vortex that coincides with the boundary of their union; similary with a larger number of adjoining loops (Fig. 2). Note that even though the

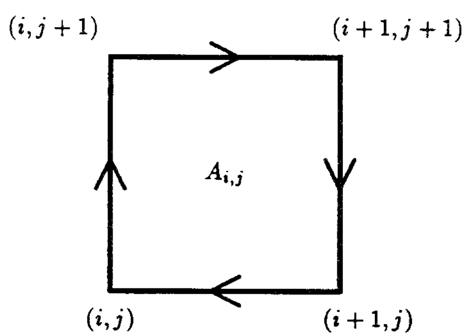

Fig. 1. An elementary vortex loop.

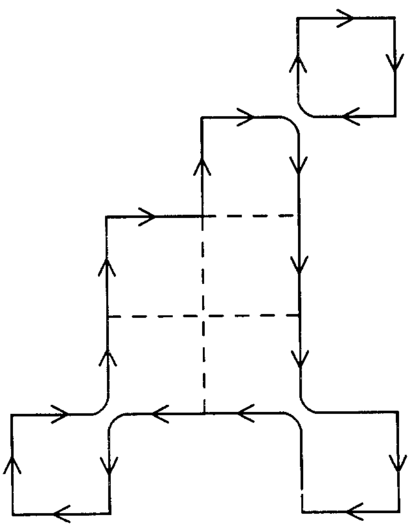

Fig. 2. Elementary vortex loops coalesce.

"vorticity" field thus constructed lies in the plane, the resulting "velocity" field, assumed given by the Biot–Savart law, is three-dimensional. In particular, two-dimensional Kosterlitz–Thouless analysis[11] does not apply. We shall henceforth omit the quotation marks around the word "vorticity" and refer to a lattice bond that contains a vortex segment as a vortex leg.

Go from square to square on the lattice, and with probability p place a microscopic vortex on each square, and with probability $1-p$ leave the square empty. Trace out the macroscopic vortex loops, which are the boundaries of connected blocks of occupied squares. Note that if an empty square is surrounded by occupied squares, the resulting macriscopic vortex is oriented anticlockwise, and thus, even though the microscopic vortices have a single orientation, the resulting macroscopic ones can have either one.

If there are vortex loops at $A_{i,j}$, $A_{i+1,j+1}$ while $A_{i+1,j}$, $A_{i,j+1}$ are empty, there arises an ambiguity as to how the macroscopic vortices are to be connected. We adopt the following convention: if $(i+j)$ is even, a loop at $A_{i,j}$ is connected to the loops that may be present at $A_{i+1,j+1}$ and $A_{i-1,j-1}$ (i.e., to the northwest and southeast) and not connected in the two other diagonal directions. If $(i+j)$ is odd, the possible connections are to the northeast and southwest (Fig. 2).

The first question to ask is the following: In what values of p can one have an infinitely long macroscopic vortex? (If such a vortex exists, we say that we have vortex percolation.) An infinitely long vortex surrounds an infinite cluster of connected occupied squares. With the rules we have set up, the existence of such a cluster is a site percolation problem on the lattice of Fig. 3, and by a standard transformation[12] becomes a bond percolation problem on a square lattice. Thus the infinite cluster exists almost surely for $p > \frac{1}{2}$. In addition, for a vortex line to be infinite, the complement of the set it surrounds must be infinite, and this happens for $p < \frac{1}{2}$. Strictly speaking, vortex percolation never happens, but at $p = \frac{1}{2}$ one can have vortex lines of arbitrary lengths. Similar situations are described in ref. 12. In Table I we exhibit the average vortex lengths in a 40×40 lattice as a function of p; the results are symmetric around $p = \frac{1}{2}$, and the average length diverges at $p = \frac{1}{2}$ when n increases. We shall refer to $p = \frac{1}{2}$ as an approximate percolation point, or a percolation point for short.

This conclusion can be expressed in terms of a temperature T. Let $\mu > 0$ be an energy associated with a microscopic vortex loop. For a system with few loops, μ is the chemical potential. The probability of a loop being present in a thermal equilibrium at a temperature T is $e^{-\mu/T}/(1 + e^{-\mu/T}) = p$, and thus $p = \frac{1}{2}$ corresponds to $T = \infty$. Percolation is approached at an infinite temperature as in the turbulent vortex case discussed in the introduction. For $T < 0$ a random system does not describe well the distribution of smooth vortices, and for $T > 0$ there are no infinitely long vortices in either the turbulent case or in the present model.

Note that at $p = \frac{1}{2}$ the entropy of the lattice vortex system is maximum. It is tempting to associate this observation with the observation in refs. 7 and 13 that the entropy of a vortex system is maximum at the percolation threshold. One should, however, note that the system considered here and the one in refs. 7 and 13 are different: here one considers the states of a fixed lattice, while the in the other papers one considers the states of a

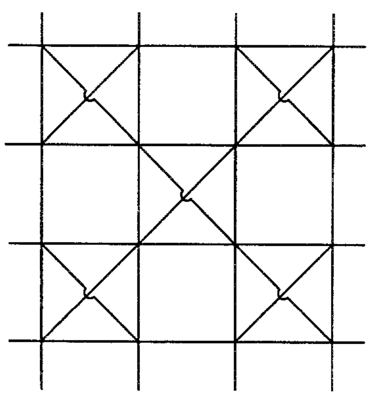

Fig. 3. Equivalent site percolation problem.

Table I. Average Vortex Length as a Function of the Probability p, 40×40 lattice

p	Average length
0.35	11.485 ± 0.005
0.40	15.48 ± 0.01
0.45	22.68 ± 0.01
0.5	32.29 ± 0.002

self-avoiding walk of a given length, with bonds not on the walk not being considered as parts of the system. The two maximum-entropy statements may be related, but the relation is not clear.

Note that at $p = \frac{1}{2}$ ($T = \infty$) large macroscopic vortices can appear and look coherent, even though the system has no long-range correlations. In Fig. 4 we exhibit some vortex lines generated by the model, with vortex loops of less than 8 legs deleted. This figure illustrates the statement that there may well be much less coherence in turbulence than meets the eye. There are indeed mechanisms in fluid mechanics that create coherence (see,

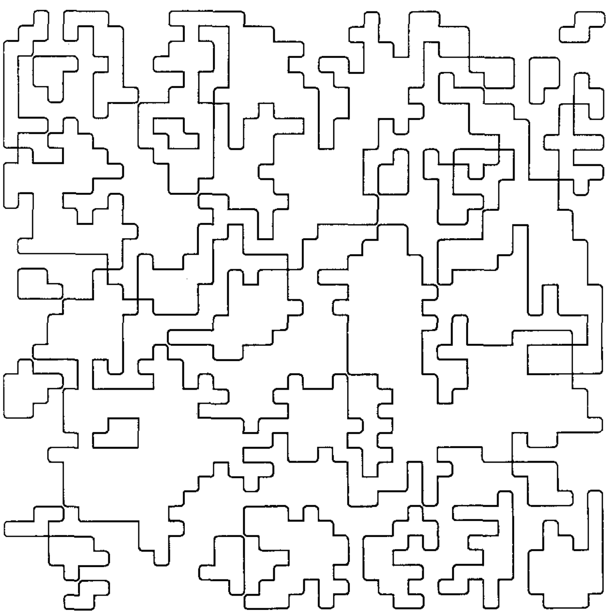

Fig. 4. Vortex lines at percolation.

Table II. Flory Exponent in Independent-Loop Model

n	θ	$\bar{N}$
20	0.603 ± 0.01	68.9 ± 0.03
40	0.612 ± 0.008	217.9 ± 0.2
60	0.612 ± 0.008	438.6 ± 0.3

e.g., ref. 6), but the resulting order may well be weak. For discussions of this coherence/disorder issue in the context of a boundary layer (often thought of as partly coherent) see, e.g., refs. 14 and 15.

Near the $p = \frac{1}{2}$ percolation point, one can readily evaluate the exponent θ for a long vortex. All one has to do is locate the intersection of the vortex with the sides of an $n \times n$ box, calculate the distance r_N between these points and the number N of legs of the vortex between them, and carry out the appropriate averaging. In this algorithm, N is variable; since the average of a log is not the log of the average, one has to be careful to caculate $\langle r_N \rangle$ for each N, then calculate $\log\langle r_N \rangle / \log N$, and only then average over N. In addition, data points that come from N small $(N < n)$ are disregarded. We omit all the other details of how percolating vortices are found; all the necessary information can be found in Stauffer.[16]

In Table II we list calculated values of θ in lattices of various sizes. We also list the average number $\bar{N}$ of legs in the percolating vortices. $\bar{N}$ increases faster than n, since the percolating vortex is fractal. The calculations turn out to be quite costly since the algorithm, by construction, requires $O(\bar{N}^{\beta})$ operations to generate a new vortex, with $\beta > 1$. The conclusion is that there is a θ independent of $\bar{N}$, with θ near 0.6, amazingly, and probably fortuitously, close to the Flory three-dimensional value. Thus, even the present independent-loop model exhibits some of the structure of the three-dimensional vortex statistics problem.

3. THE ENERGY OF A VORTEX SYSTEM

In order to prepare for the next model, we present a brief discussion of the energy of a vortex system, following refs. 7 and 13.

Consider a collection of vortex tubes in an unbounded region. Its energy is

$$E = \frac{1}{2} \int \mathbf{u}^2 \, d\mathbf{x} = \frac{1}{8\pi} \int d\mathbf{x} \int d\mathbf{x}' \, \frac{\boldsymbol{\xi}(\mathbf{x}) \cdot \boldsymbol{\xi}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

where $\mathbf{u}$ is the velocity and $\xi = \text{curl}\, \mathbf{u}$ is the vorticity. Suppose the support of ξ consists of l cylinders I_i, $i = 1, l$, arranged so as to approximate a finite number of closed loops, with ξ approximately constant in each tube and parallel to its axis. Then

$$E \cong \frac{1}{8\pi}\left(\sum_i \sum_{j \neq i} E_{ij} + \sum_i E_{ii}\right)$$

where

$$E_{ij} = \int_{I_i} d\mathbf{x} \int_{I_j} d\mathbf{x}' \frac{\xi(\mathbf{x}) \cdot \xi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Suppose all the tubes have circulation 1. Let $\mathbf{t}_i$ be a vector on the axis of I_i, in the same direction as ξ in I_i, with $|\mathbf{t}_i| =$ length of I_i. Let $|i-j|$ denote a distance between I_i and I_j. If $|i-j| \gg \max(|\mathbf{t}_i|, |\mathbf{t}_j|)$, one has

$$E_{ij} \cong \frac{\mathbf{t}_i \cdot \boldsymbol{t}_j}{|i-j|}$$

Accept this approximation whenever $i \neq j$ (this is reasonable whenever l is large). Drop the immaterial factor $(8\pi)^{-1}$. Then

$$E \cong \sum_i \sum_{j \neq i} \frac{\mathbf{t}_i \cdot \boldsymbol{t}_j}{|i-j|} + \sum_i E_{ii}$$

E_{ii} is a function of the radius r of I_i, $dE_{ii}/dr < 0$, $\lim_{r \to 0} E_{ii} = \infty$. The scaling laws obeyed by $E_{ii}(r, |\mathbf{t}_i|)$ have been examined in ref. 7. Assume all the I_i have the same radius r and the same length; then $E_{ii} = \text{const} = q$, and

$$E \cong \sum_i \sum_{j \neq i} \frac{\mathbf{t}_i \cdot \mathbf{t}_j}{|i-j|} + lq \tag{1}$$

A choice of q is a choice of r if $|\mathbf{t}_i|$ is given. A choice of q determines the energy of a microscopic vortex (its chemical potential when there are few loops), assuming (1) holds:

$$\mu = -4 + 4q$$

In this approximation, $\mu > 0$ implies $q > 1$.

4. A LATTICE MODEL WITH 3×3 INDEPENDENT BLOCKS

The next model we consider is one in which the loops interact and the configurations have a probability that depends on the energy of the vortex system. A general model of this type (equivalent to a three-dimensional XY model) will be considered elsewhere. For the present, we shall be content with a model in which the vortices remain in the plane and the loops interact in 3×3 blocks, different blocks being independent. Even this simple problem is not computationally inexpensive.

Consider a 3×3 block of loops, and the resulting vortex lines. The energy of a configuration C of the block is given by (1); if one denotes the double sum by $E_1 = E_1(C)$, the energy is $E_1(C) + l(C)\,q$, where $l = l(C)$ is the number of vortex legs. Write $q = q_0 + Q$, where $q_0 = -\min_C(E_1(C)/l(C)) = 1.12449$. Then $Q \geqslant 0$ implies $E \geqslant 0$ for all the 512 configurations C. The partition function for the 3×3 system is

$$Z = \sum_C \exp\{-\beta[E_1(C) + l(C)(q_0 + Q)]\}, \qquad \beta = T^{-1}$$

and the chemical potential is $\mu = 4(q_0 + Q) - 4$. The probability of a given configuration C is

$$P(C) = Z^{-1} \exp\{-\beta[E_1 + l(q_0 + Q)]\} \tag{2}$$

One can readily check that if $\beta = 0$, when all configurations are equally likely, $\bar{l} \equiv \sum l(C)\,P(C) = 12$. Given $\beta \neq 0$, $P(C)$ is a function of Q; the equation $\bar{l} = \bar{l}(\beta, Q) = 12$ can be solved and yields the dashed curve in Fig. 5. This curve and the Q axis divide the (β, Q) plane into the four regions I, II, III, and IV. In I and III, $\bar{l}(\beta, Q) > 12$; in II and IV, $\bar{l}(\beta, Q) < 12$. (Note that as β varies from 2.5 to 0, and then to -2.5, T varies from 0.4 to ∞, and then from $-\infty$ to -0.4. Since a positive temperature is "colder" then a negative temperature, T is monotonically increasing.) The heuristic meaning of Fig. 5 is quite clear: an increase in Q, i.e., in the cost of produc-

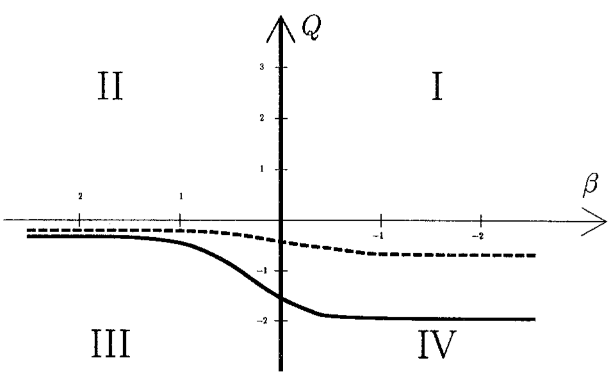

Fig. 5. Percolation loci in block model.

ing vortex legs, decreases the average number of legs if $T > 0$, and the converse is true when $T < 0$. On the other hand, $\bar{l}$ is larger than it would be with $E_1 = 0$, thus vortex folding, i.e., the creation of counterrotating nearby loops, can make up to some extent for an increase in Q; see the discussion in refs. 7 and 13, and the Kosterlitz–Thouless analysis.[2,3]

We now construct a lattice model with interaction by picking the lattice size n to be a multiple of 3, and then in each 3×3 block picking one of the 512 configurations with the probability (2), two different blocks being independent. The natural conjecture is that there will be percolation near the curves $\bar{l} = 12$ of the 3×3 blocks, since if there are too few vortex legs there will not be enough of them to make up a long vortex, and if there are too many they will join locally into smaller structures. We already know that percolation occurs at $\bar{l} = 12$ for the independent-loop model. The percolation points are not likely to coincide exactly with the curves $\bar{l} = 12$ because the overall lattice is not truly the union of the blocks, in particular since overlaps can occur at their edges. The solid curve in Fig. 5 and the Q axis are the percolation loci in the (β, Q) plane. One can thus go through a percolation state by going from positive to negative T (as in the last section) or by reducing Q. Either motion in the (β, Q) plane increases the density of vortex legs and at an appropriate point the legs coalesce into long vortices. The fact that the superfluid percolation curve is below the β axis (and thus the energy of some configurations is negative) is an artifact of our model, which, by excluding long-range interactions, emphasizes configuration of low energy. The percolation locus can be moved up by changing the boundary conditions in the 3×3 blocks. There may possibly be some significance in the fact that the curve is sloping downward. It may be that the most physical part of that percolation locus is on the left, where the temperature is low.

One can thus reach percolation (and a critical transition) by reducing the temperature from negative to positive, or by reducing the cost of

Table III. The Flory Exponent θ along the Percolation Loci

β	Q	θ
2.5	−0.1069	0.63 ± 0.01
2.0	−0.1512	0.62 ± 0.01
1.5	−0.2314	0.62 ± 0.01
0.5	−0.9067	0.60 ± 0.01
0	Indeterminate	0.61 ± 0.01
−1.0	−1.8660	0.69 ± 0.01
−2.0	−1.8521	0.70 ± 0.01

producing vortex legs. The latter can be done either by reducing Q or, at low enough Q, by varying T. The second option is more reasonable at low T. One can plausibly identify the percolation on the line $\beta = 0$ with turbulent percolation and the percolation along the nearly horizontal curve with superfluid percolation.

In Table III we display some values of θ calculated along the horizontal percolation curve. (Along $\beta = 0$ the configurations we generate are indistinguishable from the ones of the independent-loop case and θ is constant.) The variation of θ for $\beta \geqslant 0$ is slow, and may be due to statistical error. These results do make it plausible to assume that the Flory exponent varies little along the percolation loci for $\beta \geqslant 0$ and thus that is reasonable to use the same θ in the XY model and in turbulence theory.

5. CONCLUSIONS

The simple plane model of vortex filaments exhibits a percolation structure similar to what had been observed in turbulence modeling and in superfluidity. In particular, it makes plausible the idea that at the inertial range of turbulence and near the critical point of the XY model the fractal structure of the vortex filaments is similar.

In one important respect, the model is more realistic than the models in earlier work: as can be seen from Fig. 4, the model allows for a dense rather than only a sparse "suspension" of vortex lines. This added element of reality, albeit with simplified interaction, suggests that percolation models can provide a useful tool for the study of dense collections of polymers, vortices, and other stringlike objects.

Most importantly, the results presented here suggest that the analogy between the inertial range and the neighborhood of the λ point presented in ref. 8 has substance. The differences between the two are great; in particular, in a turbulent fluid, the temperature is determined by vortex stretching, varies in time, and is generally negative, while in a superfluid, vortex stretching is not a major factor (and thus Euler's equations do not apply), and the temperature is externally imposed and positive. On the other hand, the geometry of folding and screening may well be very similar and real-space renormalization group techniques may be applicable in similar ways.

ACKNOWLEDGMENT

This work was completed at the Institute for Advanced Study at Princeton, whose hospitality is gratefully acknowledged.

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